Solutions to Chapter 2

1. Calculating with the binomial probability law, for the given random variable X, we obtain

$$
\begin{aligned}
P[X &= 1] = \sum_{k=0}^{2} \binom{10}{k} (.3)^k (0.7)^{10-k} \approx 0.383, \\
P[X &= 2] = \sum_{k=3}^{5} \binom{10}{k} (0.3)^k (0.7)^{10-k} \approx 0.57, \\
P[X &= 3] = \sum_{k=6}^{8} \binom{10}{k} (0.3)^k (0.7)^{10-k} \approx 0.0472, \quad \text{and} \\
P[X &= 4] = \sum_{k=9}^{10} \binom{10}{k} (0.3)^k (0.7)^{10-k} \approx 1.44 \times 10^{-4}.
\end{aligned}
$$

The CDF is plotted in Fig. 1.

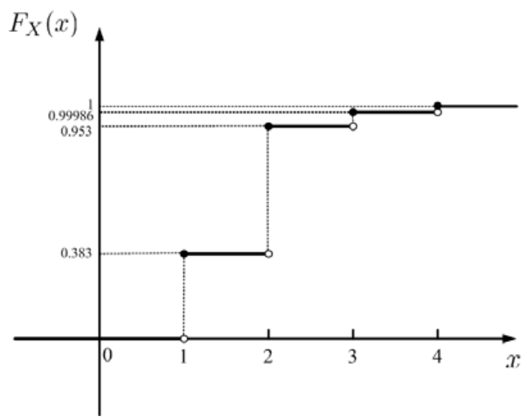


Figure 1:

2. Consider any continuous random variable. Let the outcomes be the values themselves, i.e. the random variable is an identity mapping. Then the probability of any outcome $x = \zeta$ is 0. Strictly speaking, we mean the singleton events $\{x\}$ have probability zero.

3. The cumulative distribution function (CDF) for the waiting time X is defined over $[0, \infty)$ and given as

$$
F_X(x) = \begin{cases} (x/2)^2, & 0 \le x < 1, \\ x/4, & 1 \le x < 2, \\ 1/2, & 2 \le x < 10, \\ x/20, & 10 \le x < 20, \\ 1, & 20 \le x. \end{cases}
$$

   (a) Plotting we get Fig. 2.

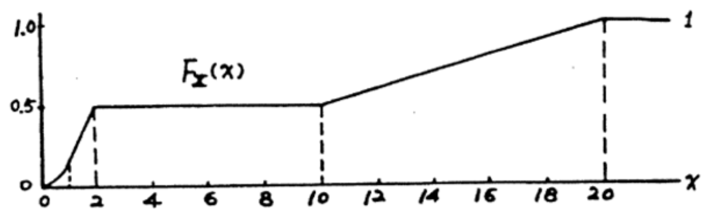


Figure 2:

(b) Taking the derivative, we get the probability density function (pdf) as

$$f_X(x) = \begin{cases} x/2, & 0 \leq x < 1, \\ 1/4, & 1 \leq x < 2, \\ 0, & 2 \leq x < 10, \\ 1/20, & 10 \leq x < 20, \\ 0, & 20 \leq x. \end{cases},$$

with sketch given as:

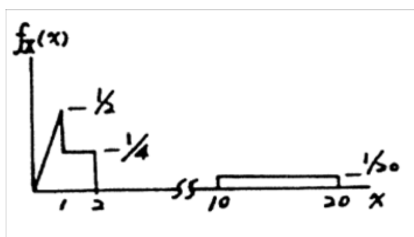


Figure 3:

We notice that

$$\begin{aligned} \int_0^\infty f_X(x)dx &= \int_0^1 \frac{x}{2}dx + \int_1^2 \frac{1}{4}dx + \int_{10}^{20} \frac{1}{20}dx \\ &= \frac{1}{4} + \frac{1}{4} + \frac{10}{20} = 1. \end{aligned}$$

(c) We compute the following probabilities:

i. [label=(0)]
ii. $P[X \geq 10] = 1 - P[X < 10] =$ [1] $1 - F_X(10) = 1/2$,
iii. $P[X < 5] = F_X(5) = 1/2$,
iv. $P[5 < X < 10] = \int_5^{10} f_X(x)dx = 0$, and
v. $P[X = 1] = 0$.

4. The point of this problem is to be careful about whether the end-points $X = b$ and $X = a$ are included in the event or not. Remember $F_X(x) \triangleq P[X \leq x]$ which includes its end-point

[1] Note: $F_X(10) = P[X \leq 10]$, but here the probability that $X = 10$ is zero.

at the right. Thus to compute $P[X < x]$, we must subtract from $F_X(x)$, the probability that $X = x$, i.e. $P[X = x]$.

$$\begin{aligned}
P[X < a] &= F_X(a) - P[X = a], \\
P[X \leq a] &= F_X(a), \\
P[a \leq X < b] &= F_X(b) - F_X(a) - P[X = b] + P[X = a], \\
P[a \leq X \leq b] &= F_X(b) - F_X(a) + P[X = a], \\
P[a < X \leq b] &= F_X(b) - F_X(a), \\
P[a < X < b] &= F_X(b) - F_X(a) - P[X = b].
\end{aligned}$$

5. For normalization, we integrate the probability density function (pdf) over the whole range of the random variable and equate it to 1.

   (a) *Cauchy distribution.* The pdf of a Cauchy random variable is given by

$$f_X(x) = \frac{B}{1 + [(x - \alpha)/\beta]^2},$$

   for $\alpha < \infty$, $\beta > 0$, and $-\infty < x < \infty$. For the integration, we will substitute $\tan\theta = \frac{x-\alpha}{\beta}$, and so $d\theta = \frac{1}{\beta}\frac{1}{1+[(x-\alpha)^2/\beta]^2}$. Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\,dx &= \int_{-\pi/2}^{\pi/2} B\beta\,d\theta \\
&= B\beta\left(\theta|_{-\pi/2}^{\pi/2}\right) \\
&= B\beta\pi \\
&= 1.
\end{aligned}$$

   Therefore, $B = \frac{1}{\beta\pi}$.

   (b) *Maxwell distribution.* The pdf of a Maxwell random variable is given by

$$f_X(x) = \begin{cases} Bx^2 e^{-x^2/\alpha^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}. \qquad (1)$$

   Integrating the pdf (substituting $y = x/\alpha$), we obtain

$$\begin{aligned}
\int_0^{\infty} Bx^2 e^{-x^2/\alpha^2}\,dx &= \int_0^{\infty} B\alpha^2 y^2 e^{-y^2}\alpha\,dy \\
&= B\alpha^3\left[\frac{-1}{2}ye^{-y^2}\Big|_0^{\infty} + \frac{1}{2}\int_0^{\infty} e^{-y^2}\,dy\right] \\
&= 0 + \frac{B\alpha^3}{2}\frac{\sqrt{\pi}}{2} \\
&= 1.
\end{aligned}$$

   Therefore, $B = \frac{4}{\sqrt{\pi}\alpha^3}$.

6. (a) *Beta distribution.* The pdf of the Beta distribution is given by

$$f_X(x) = \begin{cases} Bx^b(1-x)^c & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using the formula 6.2-1 in *Handbook of Mathematical Functions* by Abramowitz and Stegun, we get

$$\begin{aligned} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx &= \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt \\ &= 2\int_0^{\pi/2} (\text{sin}t)^{2\alpha-1}(\text{cos}t)^{2\beta-1}\,dt. \end{aligned}$$

Now, we look at the product of the Gamma function $\Gamma(x)$ evaluated at $b+1$ and $c+1$. Substitutions used for this integration are $t = v^2, u = w^2$ and $v = r\text{sin}\,\theta, w = r\text{cos}\,\theta$.

$$\begin{aligned} &\Gamma(b+1)\Gamma(c+1) \\ &= \left[\int_0^\infty t^b e^{-t}\,dt\right]\left[\int_0^\infty u^c e^{-u}\,du\right] \\ &= \int_0^\infty \int_0^\infty t^b u^c e^{-(t+u)}\,du\,dt \\ &= 4\int_0^\infty \int_0^\infty v^{2b+1} w^{2c+1} e^{-[(v^2+w^2)]}\,dv\,dw \\ &= 4\int_0^{\pi/2} \int_0^\infty r^{2b+2c+2}(\sin\,\theta)^{2b+1}(\cos\,\theta)^{2c+1} e^{-r^2}\,r dr\,d\theta \\ &= 2\int_0^\infty e^{-r^2}(r^2)^{b+c+1} 2r\,dr \int_0^{\pi/2} (\sin\,\theta)^{2b+1}(\cos\,\theta)^{2c+1}\,d\theta \\ &= \Gamma(b+c+2)2\int_0^{\pi/2} (\sin\,\theta)^{2b+1}(\cos\,\theta)^{2c+1}\,d\theta. \end{aligned}$$

Therefore, $B = \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)}$.

(b) *Chi-square distribution.* The pdf of a Chi-square random variable is given by

$$f_X(x) = \begin{cases} Bx^{(n/2)-1}e^{-x/2\sigma^2} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Integrating $f_X(x)$, we get

$$\begin{aligned} \int_0^\infty f_X(x)\,dx &= \int_0^\infty Bx^{(n/2)-1}e^{-x/2\sigma^2}\,dx \\ &= \int_0^\infty B(2\sigma^2)^{(n/2)}[(x/2\sigma^2)^{(n/2)}]e^{-x/2\sigma^2}\,dx \\ &= B(2\sigma^2)^{(n/2)}\int_0^\infty [(x/2\sigma^2)^{(n/2)}]e^{-x/2\sigma^2}\,dx \\ &= B(2\sigma^2)^{(n/2)}\Gamma\left(\frac{n}{2}\right) \\ &= 1. \end{aligned}$$

Therefore, $B = \frac{1}{(2\sigma^2)^{(n/2)}\Gamma\left(\frac{n}{2}\right)}$.

7. Here we do calculations with the Normal (Gaussian) random variable of mean 0 and given variance σ^2. In notation we often indicate this as $X : N(0,\sigma^2)$. In order to calculate the probabilities $P[|X| \geq k\sigma]$ for integer values $k = 1, 2, \ldots$, we need to convert this to the standard Normal curve that is distributed as $N(0,1)$. In particular the so-called error function is defined as

$$erf(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp(-\frac{1}{2}v^2)dv, \text{ for } x \geq 0,$$

and so only includes the right hand side of the $N(0,1)$ distribution. Expanding $P[|X| \geq k\sigma]$ for k positive, we get $P[|X| \geq k\sigma] = P[\{X \leq -k\sigma\} \cup \{X \geq k\sigma\}]$, which is somewhat cumbersome, so instead we consider the complementary event $\{|X| < k\sigma\}$ which satisfies $P[|X| \geq k\sigma] = 1 - P[|X| < k\sigma]$. For this complementary event, we have

$$\begin{aligned} P[|X| &< k\sigma] = P[-k\sigma < x < k\sigma], \quad \text{for } k > 0, \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-k\sigma}^0 \exp(-\frac{x^2}{2\sigma^2})dx + \frac{1}{\sqrt{2\pi}} \int_0^{k\sigma} \exp(-\frac{x^2}{2\sigma^2})dx \\ &= 2\frac{1}{\sqrt{2\pi}\sigma} \int_0^{k\sigma} \exp(-\frac{x^2}{2\sigma^2})dx, \text{ by the symmetry about } v = 0. \end{aligned}$$

By making the change of variable $y = x/\sigma$, we then convert this equation into

$$\begin{aligned} P[|X| &< k\sigma] = 2\frac{1}{\sqrt{2\pi}} \int_0^k \exp(-\frac{y^2}{2})dy, \quad \text{since } dy = \frac{dx}{\sigma}, \\ &= 2\,\text{erf}(k), \quad \text{for } k > 0, \end{aligned}$$

allowing us to use the standard Table 2.4-1 for $\text{erf}(\cdot)$. Looking up this value and subtracting twice it from one, we get

$$\begin{aligned} k = 1, &\quad P[|X| \geq \sigma] \doteq 0.3174, \\ k = 2, &\quad P[|X| \geq 2\sigma] \doteq 0.0456, \\ k = 3, &\quad P[|X| \geq 3\sigma] \doteq 0.0026, \\ k = 4, &\quad P[|X| \geq 4\sigma] \doteq 0.0008 \approx 0. \end{aligned}$$

8. The pdf of the Rayleigh random variable is given by

$$f_X(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} u(x).$$

Note that since $f_X(x)$ is zero for negative x, $F_X(x) = 0$, for $x < 0$. Now $F_X(k\sigma) = \int_0^{k\sigma} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}\, dx$. Substituting $y = \frac{x^2}{2\sigma^2}$ and $dy = \frac{x}{\sigma^2}$, we get

$$F_X(k\sigma) = \int_0^{k^2/2} e^{-y}\, dy = 1 - e^{-k^2/2} \qquad k = 0, 1, 2, \ldots.$$

9. For the Bernoulli random variable X, with $P_X(0) = p, P_X(1) = q$, and $q \triangleq 1 - p$, the pdf is given as

$$f_X(x) = p\delta(x) + q\delta(x-1).$$

For the binomial random variable B with parameters n and p, we have as a function of b,

$$f_B(b) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(b-k).$$

For the Poisson case, with mean $\mu_X = a$, we have the density

$$f_X(x) = \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{-a} \delta(x-k).$$

10. This problem does some calculations with a mixed random variable. We can represent the pdf of X as

$$f_X(x) = Ae^{-x}[u(x-1) - u(x-4)] + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3).$$

(a) To find the constant A, we must integrate the pdf over all x to get 1.

$$\begin{aligned} A\int_1^4 e^{-x}dx + \frac{1}{4}\int_{-\infty}^{+\infty} \delta(x-2)dx + \frac{1}{4}\int_{-\infty}^{+\infty} \delta(x-3)dx &= 1, \\ A(e^{-1} - e^{-4}) + \frac{1}{4} + \frac{1}{4} &= 1, \end{aligned}$$

which has solution $A = \frac{1}{2}\frac{1}{e^{-1}-e^{-4}} \doteq 1.43$.

(b) Taking the running integral $\int_{-\infty}^{x} f_X(v)dv$, we get the CDF $F_X(x)$ with sketch given in Fig. 4.

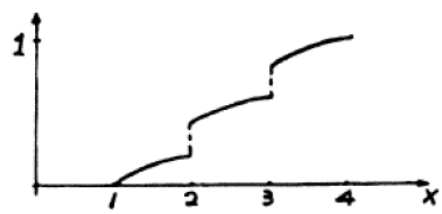


Figure 4:

Although not requested, the CDF is given analytically as

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}\frac{e^{-1}-e^{-x}}{e^{-1}-e^{-4}}, & 1 \le x < 2, \\ \frac{1}{2}\frac{e^{-1}-e^{-x}}{e^{-1}-e^{-4}} + \frac{1}{4}, & 2 \le x < 3, \\ \frac{1}{2}\frac{e^{-1}-e^{-x}}{e^{-1}-e^{-4}} + \frac{1}{2}, & 3 \le x < 4, \\ 1, & 4 \le x. \end{cases}$$

(c) We calculate the pdf as

$$f_X(x) = \frac{1}{2}\frac{e^{-x}}{e^{-1}-e^{-4}}[u(x-1) - u(x-4)] + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3).$$

So

$$\begin{aligned} P[2 \le X < 3] &= \int_2^3 \frac{1}{2}\frac{e^{-x}}{e^{-1}-e^{-4}}dx + \frac{1}{4}\int_{2^-}^{3} \delta(x-2)dx \\ &= \frac{1}{2}\frac{e^{-2}-e^{-3}}{e^{-1}-e^{-4}} + \frac{1}{4}, \end{aligned}$$

where we start the integral of the impulse at 2^- in order to pick the probability mass at $x = 2$. Note that we must include the probability mass at $x = 2$ because the event $\{2 \leq X < 3\}$ includes this point.

(d) We calculate

$$\begin{aligned} P[2 \ < \ X \leq 3] &= \int_2^3 \frac{1}{2}\frac{e^{-x}}{e^{-1}-e^{-4}}dx + \frac{1}{4}\int_2^{3^+} \delta(x-3)dx \\ &= \frac{1}{2}\frac{e^{-2}-e^{-3}}{e^{-1}-e^{-4}} + \frac{1}{4}, \end{aligned}$$

where we end the integral of the impulse at 3^+ to pick up the probability mass at $x = 3$.

(e) We have

$$\begin{aligned} &F_X(3) \\ &= P[X \leq 3] \\ &= \int_1^3 \frac{1}{2}\frac{e^{-x}}{e^{-1}-e^{-4}}dx + \frac{1}{4}\int_1^{3^+} \delta(x-3)dx + \frac{1}{4}\int_1^{3^+} \delta(x-2)dx \\ &= \frac{1}{2}\frac{e^{-1}-e^{-3}}{e^{-1}-e^{-4}} + \frac{1}{4} + \frac{1}{4}. \end{aligned}$$

11. First we need to calculate the probability that X is less than 1 and that it is greater than 2 (area of shaded region in Fig. 5). Now

$$P[X < 1] = P[X \leq 1] = F_X(1) = 1 - e^{-1}.$$

$$P[X > 2] = 1 - P[X \leq 2] = 1 - F_X(2) = 1 - (1 - e^{-2}) = e^{-2}.$$

Since the events are disjoint, the probability that $X < 1$ or $X > 2$ is

$$P[\{X < 1\} \cup \{X > 2\}] = P[X < 1] + P[X > 2] = 1 - e^{-1} + e^{-2} = 0.767.$$

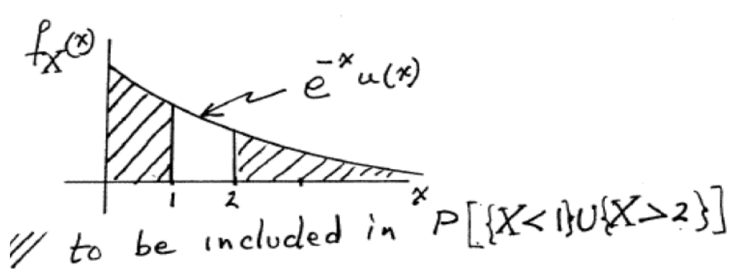


Figure 5:

12. We calculate the pdf as

$$f_X(x) = Ae^{-x}\left[u(x-1) - u(x-4)\right] + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3).$$

So

$$\begin{aligned} P[2 &< X < 4] = \int_2^4 Ae^{-x}dx + \frac{1}{4} \\ &= A(e^{-2} - e^{-4}) + \frac{1}{4}, \end{aligned}$$

where we start the integral of the impulse at 2^+ in order to not include the probability mass at $x = 2$. The overall answer then becomes $1.43(e^{-2} - e^{-4}) + 0.25$.

13. This is an example where the probability distribution is defined on the sample space which is not the elementary sample space. Normally, when we consider two coins tossed simultaneously, we consider the sample space containing two tuples of heads and tails, indicating the outcome of two tosses, i.e., we consider the sample space $\Omega = \{HH, HT, TH, TT\}$. Here we will see that we can also define probability on another set of outcomes.

    The sample space Ω contains outcomes $\zeta_1, \zeta_2, \zeta_3$ that denote outcomes of two, one, and no heads, respectively. Assuming that the coins are unbiased, we first find the probability of these outcomes.
    $P[\zeta_1] = P[\text{heads on both tosses}] = 0.5 \times 0.5 = 0.25$
    $P[\zeta_2] = P[\text{head on first, tail on second}] + P[\text{tail on first, head on second}] = 0.5 \times 0.5 + 0.5 \times 0.5 = 0.5$
    $P[\zeta_3] = P[\text{tails on both tosses}] = 0.5 \times 0.5 = 0.25$

    (a) $\{\zeta : X(\zeta) = 0, Y(\zeta) = -1\} = \zeta_2 \implies P[\{\zeta : X(\zeta) = 0, Y(\zeta) = -1\}] = P[\zeta_2] = 0.5$
    $\{\zeta : X(\zeta) = 0, Y(\zeta) = 1\} = \zeta_1 \implies P[\{\zeta : X(\zeta) = 0, Y(\zeta) = 1\}] = P[\zeta_1] = 0.25$
    $\{\zeta : X(\zeta) = 1, Y(\zeta) = -1\} = \phi \implies P[\{\zeta : X(\zeta) = 1, Y(\zeta) = -1\}] = P[\phi] = 0$
    $\{\zeta : X(\zeta) = 1, Y(\zeta) = 1\} = \zeta_3 \implies P[\{\zeta : X(\zeta) = 1, Y(\zeta) = 1\}] = P[\zeta_3] = 0.25$

    (b) Before we find the independence of X and Y, we first find the probability mass functions (pmf) of X and Y.
    $P_X[0] = P[X = 0] = P[\{\zeta_1\}] + P[\{\zeta_2\}] = 0.25 + 0.5 = 0.75$
    $P_X[1] = P[X = 1] = P[\{zeta_3\}] = 0.25$.
    $P_X[k] = 0$ for $k \neq 0, 1$.

    Similarly, $P_Y[-1] = 0.5, P_Y[1] = 0.5, P_Y[k] = 0$ for $k \neq -1, 1$.
    For independence of X, Y, we need $P_{X,Y}[a, b] = P_X[a]P_Y[b]$. For $a = 0, b = 1$, $P_{X,Y}[0, 1] = 0.25, P_X[0]P_Y[1] = 0.75 \times 0.5 = 0.375$.
    Hence X and Y are not independent.

14. (a) We have to integrate the given density over the full domain. We know

$$\begin{aligned}
\int_{-\infty}^{+\infty} f_X(x)dx &= 1 \\
&= \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + A\int_{-2}^{+2} x^2 dx \\
&= \frac{1}{4} + 2A\int_{0}^{+2} x^2 dx \\
&= \frac{1}{4} + 2A\left(\frac{1}{3}x^3\Big|_0^2\right) \\
&= \frac{1}{4} + 2A\frac{8}{3}.
\end{aligned}$$

Hence $A = 9/64$.

(b) A labeled plot appears below in Fig. 6. Note the jumps occurring at the impulse locations in the density. Also note the slope of the distribution function is given by the density function in the smooth regions.

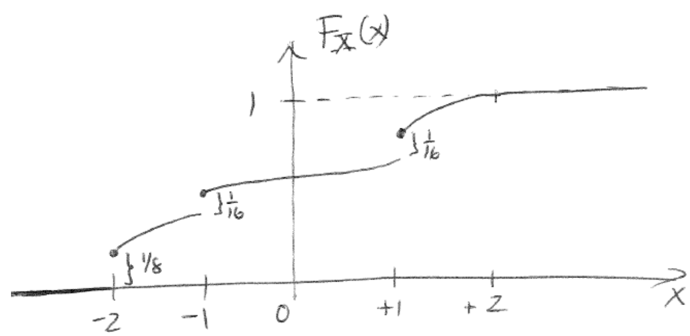


Figure 6:

(c) We proceed as follows

$$\begin{aligned}
F_X(1) &= \int_{-\infty}^{1} f_X(x)dx \\
&= \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{9}{64}\int_{-2}^{+1} x^2 dx \\
&= \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \left(\frac{3}{8} + \frac{9}{64}\int_{0}^{+1} x^2 dx\right) \\
&\qquad \text{(using the symmetry of } x^2\text{)} \\
&= \frac{1}{4} + \left(\frac{3}{8} + \frac{9}{64}\left(\frac{1}{3}x^3\Big|_0^1\right)\right) \\
&= \frac{1}{4} + \frac{3}{8} + \frac{9}{64}\frac{1}{3} = \frac{43}{64}.
\end{aligned}$$

(d) We can calculate

$$P[-1 < X \le 2] = \int_{-\infty}^{2} f_X(x)dx = \frac{1}{16} + \frac{9}{64}\int_{-1}^{+2} x^2 dx.$$

But the easier way is to realize that $\int_{-1}^{+2} x^2 dx = \int_{-2}^{+1} x^2 dx$ (because of symmetry) which was needed in part (c). Then, by just counting the one relevant impulse area, we can write

$$P[-1 < X \le 2] = \frac{1}{16} + \frac{3}{8} + \frac{3}{64} = \frac{31}{64}.$$

15. First we calculate the probability that X is even. Now it is binomial distributed with parameters $n = 4$ and $p = 0.5$, i.e. $b(k; 4, 0.5), 0 \le k \le 4$, thus

$$\begin{aligned} P[\{X = \text{even}\}] &= b(0;4,0.5) + b(2;4,0.5) + b(4;4,0.5) \\ &= \binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 \\ &= 1 \times \left(\frac{1}{2}\right)^4 + 6 \times \left(\frac{1}{2}\right)^4 + 1 \times \left(\frac{1}{2}\right)^4 = \frac{1}{2}. \end{aligned}$$

Now, the conditional probability is given as

$$P[\{X = k\}|\{X = \text{even}\}] = \frac{P[\{X = k\} \cap \{X = \text{even}\}]}{P[\{X = \text{even}\}]} = \begin{cases} 2\frac{1}{16} = \frac{1}{8}, & k = 0, \\ 0, & k = 1, \\ 2\frac{6}{16} = \frac{6}{8}, & k = 2, \\ 0, & k = 3, \\ 2\frac{1}{16} = \frac{1}{8}, & k = 4, \end{cases}$$

where we have used the fact that the joint event

$$\{X = k\} \cap \{X = \text{even}\} = \begin{cases} \{X = k\}, & k \text{ even}, \\ \phi, & k \text{ odd}. \end{cases}$$

16. The marginal distribution function of random variable N is given by

$$F_N(n) = F_{W,N}(+\infty, n) = \begin{cases} 0, & n < 0, \\ \frac{n}{10}, & 0 \le n < 5, \\ \frac{n}{10}, & 5 \le n < 10, \\ 1, & n \ge 10. \end{cases}$$

The pmf is given by

$$P_N(n) = F_N(n) - F_N(n-1) = \begin{cases} 0, & n \le 0, \\ \frac{1}{10}, & 0 < n \le 10, \\ 0, & n > 10. \end{cases}$$

The conditional probability density function:

$$\begin{aligned} P[W \le w, N = n] &= F_{W,N}(w,n) - F_{W,N}(w, n-1) \\ &= u(w) \begin{cases} 0, & n \le 0 \\ (1 - e^{-w/\mu_0})\frac{1}{10}, & 1 \le n \le 5 \\ (1 - e^{-w/\mu_1})\frac{1}{10}, & 5 < n \le 10 \\ 0, & n > 10 \end{cases}. \end{aligned}$$

Note that $F_W(w|N = n) = P[W \le w|N = n]$ is not defined for $n \le 0$ or $n > 10$, because for these n, $P[N = n] = 0$. Therefore,

$$\begin{aligned} F_W(w|N = n) &= \frac{P[W \le w, N = n]}{P_N(n)} \\ &= u(w) \begin{cases} (1 - e^{-w/\mu_0}), & 1 \le n \le 5 \\ (1 - e^{-w/\mu_1}), & 5 < n \le 10 \end{cases}. \end{aligned}$$

Hence,

$$f_W(w|N = n) = u(w) \begin{cases} \frac{1}{\mu_0} e^{-w/\mu_0}, & 1 \le n \le 5 \\ \frac{1}{\mu_1} e^{-w/\mu_1}, & 5 < n \le 10. \end{cases}$$

17. Let the number of bulbs produced by A and B be n_A and n_B respectively. We have $n_A + n_B = n$, and n is the total number of the bulbs. So $P[A] = \frac{n_A}{n} = \frac{1}{4}$ and $P[B] = \frac{n_B}{n} = \frac{3}{4}$. Since we have

$$F_X(x|A) = (1 - e^{-0.2x})u(x), \quad F_X(x|B) = (1 - e^{-0.5x})u(x),$$

then

$$\begin{aligned} F_X(x) &= F_X(x|A)P(A) + F_X(x|B)P(B) \\ &= \frac{1}{4}(1 - e^{-0.2x})u(x) + \frac{3}{4}(1 - e^{-0.5x})u(x). \end{aligned}$$

So

$$F(2) = \frac{1}{4}(1 - e^{-0.2\times2}) + \frac{3}{4}(1 - e^{-0.5\times2}) = 0.56,$$

$$F(5) = \frac{1}{4}(1 - e^{-0.2\times5}) + \frac{3}{4}(1 - e^{-0.5\times5}) = 0.85,$$

$$F(7) = \frac{1}{4}(1 - e^{-0.2\times7}) + \frac{3}{4}(1 - e^{-0.5\times7}) = 0.92.$$

Then $P[\text{burns at least 2 months}] = 1 - F(2) = 0.44$, $P[\text{burns at least 5 months}] = 1 - F(5) = 0.15$ and $P[\text{burns at least 7 months}] = 1 - F(7) = 0.08$.

18. Given the event $A \triangleq \{b < X \le a\}$, for $b < a$, we calculate $F_X(x|A)$.

i) $x \le b : F_X(x|A) = 0$, since the joint event $\{X \le b\} \cap A = \phi$.

ii) $x > a : F_X(x|A) = 1$, since the joint event $\{X \le a\} \cap A = A$, so the conditional probability of $\{X \le a\}$, given A, is one.

iii) $b < x \leq a$: Here we must calculate the actual intersection of the two sets $\{X \leq a\}$ and $A = \{b < X \leq a\}$. Since $b < x \leq a$, we get $\{X \leq a\} \cap A = \{X \leq x\} \cap \{b < X \leq a\} = \{b < X < x\}$. We can then calculate the conditional probability

$$\begin{aligned} F_X(x|A) &= \frac{P[\{X \leq x\} \cap A]}{P[A]} \\ &= \frac{P[\{b < X < x\}]}{P[A]} \\ &= \frac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)}, \quad \text{for } b < x \leq a. \end{aligned}$$

19. In order to get $P[Y = k]$, we can consider $P[Y = k|X = x]$ first and then do the integral over all x.

$$\begin{aligned} P[Y = k] &= \int_{-\infty}^{\infty} P[Y = k|X = x] f_X(x) dx \\ &= \frac{1}{5} \int_0^5 \frac{x^k e^{-x}}{k!} dx = \frac{1}{5k!} \int_0^5 x^k e^{-x} dx \end{aligned}.$$

for $k = 0$:

$$P[Y = 0] = \frac{1}{5}(1 - \frac{1}{0!} e^{-5})$$

for $k = 1$:

$$P[Y = 1] = \frac{1}{5}(1 - \frac{1}{0!} e^{-5} - \frac{5^1}{1!} e^{-5})$$

for $k = 2$:

$$P[Y = 2] = \frac{1}{5}\left(1 - \frac{1}{0!} e^{-5} - \frac{5^1}{1!} e^{-5} - \frac{5^2}{2!} e^{-5}\right)$$

for general k:

$$P[Y = k] = \frac{1}{5}\left(1 - \frac{1}{0!} e^{-5} - \frac{5^1}{1!} e^{-5} ... - \frac{5^k}{k!} e^{-5}\right), \quad k \geq 0.$$

20. (a) The pmf of X is binomial with $n = 8$ and $p = q = 0.5$, i.e.

$$P_X(k) \triangleq P[X = k] = b(k; 8, 0.5).$$

This is because the 8 votes are independent, each with $p = 0.5$ chance of being favorable. They are thus Bernoulli trials, which leads to the binomial distribution in the binary case. We note that since $p = 0.5$, the distribution will be symmetric about $X = k = 4$.

(b) We must find the conditional PDF $F_X(x|A)$ for the range $-1 \leq x \leq 10$. Now

$$\begin{aligned} F_X(x|A) &\triangleq P[X \leq x|A] = \frac{P[\{X \leq x\}\} \cap \{X > 4\}]}{P[X > 4]} \\ &= \frac{P[4 < X \leq x]}{P[X > 4]}. \end{aligned}$$

Since X is binomially distributed, we have

$$\begin{aligned} P[X > 4] &= \left(\frac{1}{2}\right)^8 \sum_{k=5}^{8} \binom{8}{k} \\ &= \frac{1}{2}\left(1 - \binom{8}{4}\left(\frac{1}{2}\right)^8\right) \\ &= \frac{93}{256}, \end{aligned}$$

where the second to last line is by symmetry of this binomial distribution about $k = 4$. Turning to the numerator, we have

$$P[4 < X \leq x] = \begin{cases} 0, & x < 5 \\ \left(\frac{1}{2}\right)^8 \sum_{k=5}^{8} \binom{8}{k} u(x-k), & x \geq 5. \end{cases}$$

Then

$$\begin{aligned} F_X(x|A) &= \begin{cases} 0, & x < 5 \\ \frac{256}{93}\left(\frac{1}{2}\right)^8 \sum_{k=5}^{8} \binom{8}{k} u(x-k), & x \geq 5. \end{cases} \\ &= \left(\frac{1}{93}\sum_{k=5}^{8}\binom{8}{k} u(x-k)\right) u(x-5) \\ &= \left(\frac{1}{93}\sum_{k=5}^{\lfloor x \rfloor}\binom{8}{k}\right) u(x-5), \end{aligned}$$

where $\lfloor x \rfloor$ denotes the least integer function

$$\lfloor x \rfloor \triangleq x \text{ rounded } \textit{down} \text{ to next integer.}$$

Calculating, we determine

$$\frac{1}{93}\binom{8}{k} = \begin{cases} \frac{56}{93}, & k = 5, \\ \frac{28}{93}, & k = 6, \\ \frac{8}{93}, & k = 7, \\ \frac{1}{93}, & k = 8, \end{cases}$$

and thus

$$F_X(x|A) = \begin{cases} \frac{56}{93}, & k = 5, \\ \frac{84}{93}, & k = 6, \\ \frac{92}{93}, & k = 7, \\ 1, & k = 8. \end{cases}$$

Now for $x < 5$, $F_X(x|A) = 0$, and for $x > 8$, $F_X(x|A) = 1$, so we have the plot of Figure 1.

(c) From the calculations done above and from the definition

$$\begin{aligned} f_X(x|A) &= \frac{dF_X(x|A)}{dx} \\ &= \frac{1}{93}\sum_{k=5}^{8}\binom{8}{k}\delta(x-k) \\ &= \frac{56}{93}\delta(x-5) + \frac{28}{93}\delta(x-6) + \frac{8}{93}\delta(x-7) + \frac{1}{93}\delta(x-8), \end{aligned}$$

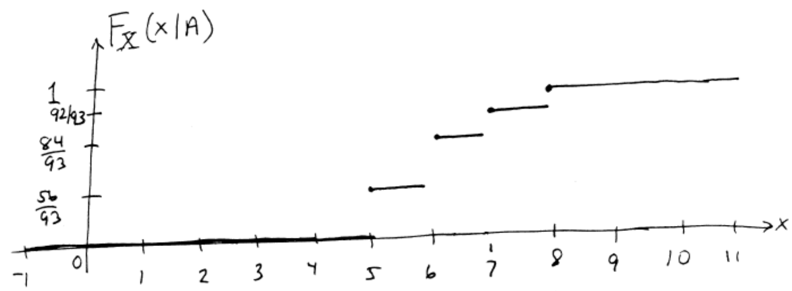


Figure 7:

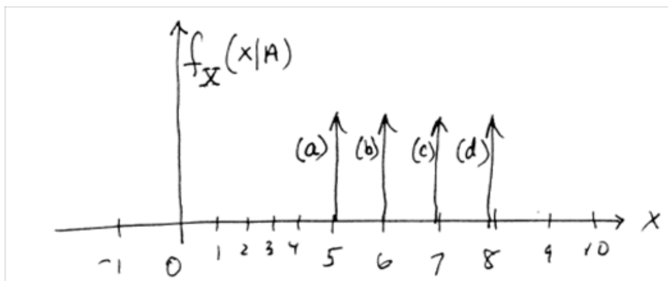


Figure 8:

with plot of Figure 2. Note we write the areas of the impulses in parentheses. In this figure, $a = 56/93$, $b = 28/93$, $c = 8/93$, and $d = 1/93$.

(d) Using Bayes' rule, we have

$$
\begin{aligned}
P[4 \leq X \leq 5|A] &= \frac{P[4 < X \leq 5]}{P[X > 4]} \\
&= \frac{P[X = 5]}{P[X > 4]} \\
&= \frac{\left(\frac{1}{2}\right)^8 \binom{8}{5}}{\frac{93}{2^8}} = \frac{1}{93}\binom{8}{5} = \frac{56}{93}.
\end{aligned}
$$

21. The random variables X and Y have joint probability density function (pdf)

$$
f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}x^2(1-y), & 0 \leq x \leq 2, 0 \leq y \leq 1, \\ 0, & \text{else.} \end{cases}
$$

(a) To find $P[X \leq 0.5]$, we start with

$$\begin{aligned} P[X &\leq 0.5] = \int_{-\infty}^{0.5}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy \\ &= \int_0^{0.5}\int_0^1 \frac{3}{4}x^2(1-y)dxdy \\ &= \frac{3}{4}\left(\int_0^{0.5} x^2dx\right)\left(\int_0^1 (1-y)dy\right) \\ &= \frac{3}{4}\left(\frac{x^3}{3}|_0^{0.5}\right)\left((y-\frac{y^2}{2})|_0^1\right) \\ &= \frac{3}{4}\frac{1}{24}(1-\frac{1}{2}) = \frac{1}{64}. \end{aligned}$$

.

(b) By definition

$$\begin{aligned} F_Y(0.5) &= P[Y \leq 0.5] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{0.5} f_{X,Y}(x,y)dxdy \\ &= \int_0^2\int_0^{0.5} \frac{3}{4}x^2(1-y)dxdy \\ &= \frac{3}{4}\left(\frac{x^3}{3}|_0^2\right)\left((y-\frac{y^2}{2})|_0^{0.5}\right) \\ &= \frac{3}{4}\frac{8}{3}\left(\frac{1}{2}-\frac{1}{8}\right) = \frac{3}{4}. \end{aligned}$$

(c) To find $P[X \leq 0.5|Y \leq 0.5]$, we note that X and Y are independent random variables, so the answer is the same as in part a), namely $P[X \leq 0.5|Y \leq 0.5] = P[X \leq 0.5] = \frac{1}{64}$. However, we can also calculate directly,

$$\begin{aligned} P[X &\leq 0.5|Y \leq 0.5] = \frac{P[X \leq 0.5, Y \leq 0.5]}{P[Y \leq 0.5]} \\ &= \int_0^{0.5}\int_0^{0.5} \frac{3}{4}x^2(1-y)dxdy/\left(\frac{3}{4}\right) \\ &= \frac{3}{4}\frac{1}{24}\left(\frac{1}{2}-\frac{1}{8}\right)/\left(\frac{3}{4}\right) = \frac{1}{64}. \end{aligned}$$

(d) Here, we can note again that X and Y are independent random variables for the given joint pdf, and thus

$$\begin{aligned} P[Y &\leq 0.5|X \leq 0.5] = P[Y \leq 0.5] \\ &= \frac{3}{4} \quad \text{from part b).} \end{aligned}$$

22. To check for independence, we need to look at the marginal pdfs of X and Y. How do we find the pdf's? We can use the property that the pdf must integrate to 1. Say $f_X(x) = Ae^{-\frac{1}{2}(\frac{x}{3})^2}u(x)$, and $\int_0^\infty f_X(x)dx = 1$, we find $A = \frac{2}{3\sqrt{2\pi}}$. Similarly, $f_Y(y) = Be^{-\frac{1}{2}(\frac{x}{2})^2}u(y)$,

and $\int_0^\infty f_Y(y)dy = 1$, so $B = \frac{2}{2\sqrt{2\pi}}$. Multiplying the two marginal pdfs, we see that the product is indeed equal to joint pdf; i.e., $f_X(x)f_Y(y) = f_{X,Y}(x,y)$. Therefore, X and Y are independent random variables; their joint probability factors and hence $P[0 < X \leq 3, 0 < Y \leq 2] = P[0 < X \leq 3]P[0 < Y \leq 2]$. Thus

$$\begin{aligned} P[0 < X \leq 3] &= \int_{-3}^{3} \frac{2}{3\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x}{3})^2} dx \\ &= 2 \times \frac{2}{3\sqrt{2\pi}} \int_0^3 e^{-\frac{1}{2}(\frac{x}{3})^2} dx = 2erf(1), \end{aligned}$$

$$\begin{aligned} P[0 < Y \leq 2] &= \int_{-2}^{2} \frac{2}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y}{2})^2} dy \\ &= 2 \times \frac{2}{2\sqrt{2\pi}} \int_0^2 e^{-\frac{1}{2}(\frac{y}{2})^2} dy = 2erf(1). \end{aligned}$$

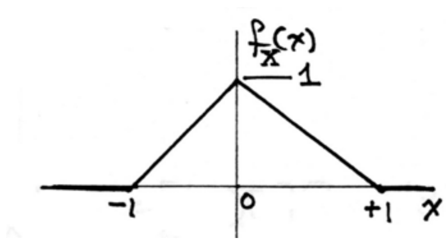
So

$$P[0 < X \leq 3, 0 < Y \leq 2] = P[0 < X \leq 3]P[0 < Y \leq 2]$$
$$= 2\,\mathrm{erf}(1) \times 2\,\mathrm{erf}(1) = 4\,\mathrm{erf}(1)^2 \doteq 0.466.$$

23. (a) Since

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} f_X(x)dx \\ &= A\int_{-1}^{0}(1+x)dx + A\int_0^{+1}(1-x)dx \\ &= A\left(x + \frac{x^2}{2}\right)\Bigg|_{-1}^{0} + A\left(x - \frac{x^2}{2}\right)\Bigg|_0^1 \\ &= A\left(\frac{1}{2} + \frac{1}{2}\right) \\ &= A. \end{aligned}$$

Thus $A = 1$ and f_X is plotted as



(b) $F_X(x) = 0$ for $x \le -1$. Then for $-1 < x \le 0$, we calculate

$$\begin{aligned} F_X(x) &= \int_{-1}^{x} (1+v)dv \\ &= \left(v + \frac{v^2}{2}\right)\Big|_{-1}^{x} \\ &= x + \frac{x^2}{2} - \left(-1 + \frac{(-1)^2}{2}\right) \\ &= x + \frac{x^2}{2} + \frac{1}{2}. \end{aligned}$$

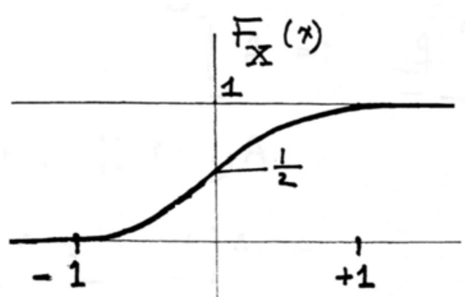
We note that $F_X(x) = \frac{1}{2}$. Then for $0 < x \le 1$, we calculate

$$\begin{aligned} F_X(x) &= \frac{1}{2} + \int_{0}^{x} (1-v)dv \\ &= \frac{1}{2} + \left(v - \frac{v^2}{2}\right)\Big|_{0}^{x} \\ &= \frac{1}{2} + x - \frac{x^2}{2}. \end{aligned}$$

Note that $F_X(x) = 1$ for $x \ge 1$ since $\int_{-1}^{+1} f_X(x)dx = 1$. Putting all the results together, we get

$$F_X(x) = \begin{cases} 0, & x \le -1, \\ \frac{1}{2} + x + \frac{x^2}{2}, & -1 < x \le 0, \\ \frac{1}{2} + x - \frac{x^2}{2}, & 0 < x \le 1 \\ 1, & x > 1. \end{cases}$$

The sketch of F_X is shown below.



(c) $P[X > b] = 1 - F_X(b) = \frac{1}{2}F_X(b)$, which gives $F_X(b) = \frac{2}{3}$, therefore $b \in (0,1)$. In this interval $F_X(b) = \frac{1}{2} + b - \frac{b^2}{2}$, so we have the quadratic equation

$$3b^2 - 6b + 1 = 0,$$

which is solved by roots $b_{1,2} = 1 \pm \sqrt{\frac{2}{3}}$. The root in $(0,1)$ is then $b = 1 - \sqrt{\frac{2}{3}} \simeq 0.185$.

24. The general expression is given as:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2\sigma^2(1-\rho^2)}(x^2 + y^2 - 2\rho xy)\right).$$

If $\rho = 0$ and $\sigma = 1$, then this becomes

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi}\exp\left(\frac{-1}{2}(x^2+y^2)\right) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \\ &= f_X(x)f_Y(y). \end{aligned}$$

The desired joint probability can be calculated as

$$\begin{aligned} P\left[-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}\right] &= P\left[-\frac{1}{2} < X \leq \frac{1}{2}\right]P\left[-\frac{1}{2} < Y \leq \frac{1}{2}\right] \\ &= 2\,\mathrm{erf}\left(\frac{1}{2}\right)2\,\mathrm{erf}\left(\frac{1}{2}\right) \\ &= \left[2\,\mathrm{erf}\left(\frac{1}{2}\right)\right]^2 \\ &\simeq 0.144. \end{aligned}$$

25. We use Bayes' formula for pdf's:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}.$$

We have

$$f_X(x) = \frac{1}{2}\mathrm{rect}(\frac{x}{2}).$$

Then

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx &= \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx \\ &= \int_{-1}^{1}\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(y-x)^2}{2\sigma^2}\right]dx. \end{aligned}$$

Let $\xi = \frac{x-y}{\sigma}$, then $d\xi = \frac{dx}{\sigma}$ and we obtain

$$f_Y(y) = \frac{1}{2}\int_{\frac{-1-y}{\sigma}}^{\frac{1-y}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2}d\xi = \frac{1}{2}\left[\mathrm{erf}(\frac{1-y}{\sigma}) - \mathrm{erf}(\frac{-1-y}{\sigma})\right].$$

But $erf(x) = -erf(-x)$, hence

$$f_Y(y) = \frac{1}{2}\left[\mathrm{erf}(\frac{1+y}{\sigma}) - \mathrm{erf}(\frac{y-1}{\sigma})\right].$$

Then finally

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{\frac{1}{\sqrt{2\pi}\ \sigma}\exp[-\frac{(y-x)^2}{2\sigma^2}]\mathrm{rect}(\frac{x}{2})}{\mathrm{erf}(\frac{1+y}{\sigma}) - \mathrm{erf}(\frac{y-1}{\sigma})}.$$

26. We start off with the general relation for conditional probability densities

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)}{f_Y(y)}.
\end{aligned}
$$

Next, we find the denominator as

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{+\infty} f_{Y,X}(y,x)dx \\
&= \int_{-\infty}^{+\infty} f_{Y|X}(y|x)f_X(x)dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)dx,
\end{aligned}
$$

and so, combining this result with the one above, we have

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)}{f_Y(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)}{\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)dx} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)}{\frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{1}{2}e^{-(y-1)^2/2\sigma^2}+\frac{1}{2}e^{-(y+1)^2/2\sigma^2}\right)} \\
&= \frac{e^{-(y-x)^2/2\sigma^2}\left(\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)\right)}{\frac{1}{2}e^{-(y-1)^2/2\sigma^2}+\frac{1}{2}e^{-(y+1)^2/2\sigma^2}}, \quad \text{or equivalently} \\
&= \frac{e^{-(y-x)^2/2\sigma^2}}{e^{-(y-1)^2/2\sigma^2}+e^{-(y+1)^2/2\sigma^2}}\left(\delta(x-1)+\delta(x+1)\right).
\end{aligned}
$$

Note, we could have eliminated a few steps in our solution by starting from the Total Probability Theorem for density functions, from which we can write directly

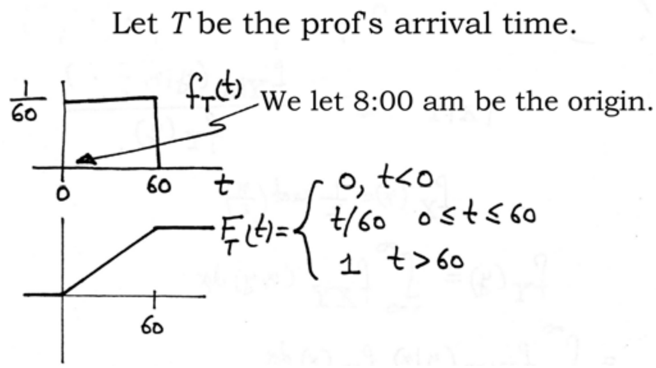
$$
f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{\int_{-\infty}^{+\infty} f_{Y|X}(y|x)f_X(x)dx}.
$$

If the question had asked instead for the conditional probability mass function (PMF) $P_{X|Y}$, the answer would have been

$$
P_{X|Y}(x|y) = \begin{cases} \frac{e^{-(y-x)^2/2\sigma^2}}{e^{-(y-1)^2/2\sigma^2}+e^{-(y+1)^2/2\sigma^2}}, & x = \pm 1, \\ 0, & \text{else} \end{cases},
$$

as can be easily obtained by integrating the conditional density found above.

27.



$$\begin{aligned} P[A] &= P[T > 30] = 1 - F_T(30) \\ P[B] &= P[T \leq 31] = F_T(31) \\ P[AB] &= P[30 < T \leq 31] = F_T(31) - F_T(30) \\ P[B|A] = \frac{P[AB]}{P[A]} &= \frac{F_T(31) - F_T(30)}{1 - F_T(30)} \\ &= \frac{\frac{31-30}{60}}{\frac{60-30}{60}} = \frac{1}{30}. \end{aligned}$$

$$\begin{aligned} P[A|B] &= \frac{P[AB]}{P[B]} = \frac{F_T(31) - F_T(30)}{F_T(31)} \\ &= \frac{\frac{31-30}{60}}{\frac{31}{60}} = \frac{1}{31}. \end{aligned}$$

28. (a)

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} f_X(x) dx \\ &= c \int_0^{\infty} e^{-2x} dx \\ &= c \left(\left. \frac{e^{-2x}}{-2} \right|_0^{\infty} \right) \\ &= c \left(0 - -\frac{1}{2} \right) \\ &= c/2, \end{aligned}$$

thus we must have $c = 2$.

(b) For $x > 0, a > 0$, we can write

$$\begin{aligned} P[X \geq x+a] &= 2\int_{x+a}^{\infty} e^{-2v}dv \\ &= 2\left(\frac{e^{-2v}}{-2}\Big|_{x+a}^{\infty}\right) \\ &= 2\left(0 - -\frac{e^{-2(x+a)}}{2}\right) \\ &= e^{-2(x+a)}, \quad \text{with } x > 0, a > 0. \end{aligned}$$

(c)

$$P[X \geq x+a|X > a] = \frac{P[X \geq x+a, X > a]}{P[X > a]},$$

but note that for $x > 0$, the event $\{X \geq x+a\}$ is a subset of the event $\{X > a\}$, so $\{X \geq x+a\} \cap \{X > a\} = \{X \geq x+a\}$, and hence $P[X \geq x+a, X > a] = P[X \geq x+a]$, thus we have, for $x > 0$,

$$\begin{aligned} P[X \geq x+a|X > a] &= \frac{P[X \geq x+a, X > a]}{P[X > a]} \\ &= \frac{P[X \geq x+a]}{P[X > a]} \\ &= \frac{e^{-2(x+a)}}{e^{-2a}} \\ &= e^{-2x}, \quad \text{independent of } a\,! \end{aligned}$$

Since this conditional probability is (functionally) independent of the variable a, the *memory* of a had been lost.

29. We need to solve for y in

$$1 - e^{-y} \geq 0.95.$$

But this implies that

$$\begin{aligned} y &\geq -\ln(0.005) \\ &\doteq 2.996. \end{aligned}$$

Thus, $y = 3$ should do.

30. This is a rather classic problem in detection theory.

$$\begin{aligned} P[A|M] &= P[X \geq 0.5|M] \\ &= \frac{1}{\sqrt{2\pi}}\int_{0.5}^{\infty} e^{-\frac{1}{2}(x-1)^2}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-0.5}^{\infty} e^{-\frac{1}{2}y^2}dy, \quad \text{with } y \triangleq x-1, \\ &= \frac{1}{2} + \text{erf}(0.5) \doteq 0.69. \end{aligned}$$

Then

$$\begin{aligned} P[A|M^c] &= P[X \geq 0.5|M^c] \\ &= \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}x^2} dx \\ &= \frac{1}{2} - \text{erf}(0.5) \doteq 0.31, \end{aligned}$$

$$\begin{aligned} P[A^c|M^c] &= P[X < 0.5|M^c] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}x^2} dx \\ &= \frac{1}{2} + \text{erf}(0.5) \doteq 0.69, \end{aligned}$$

and

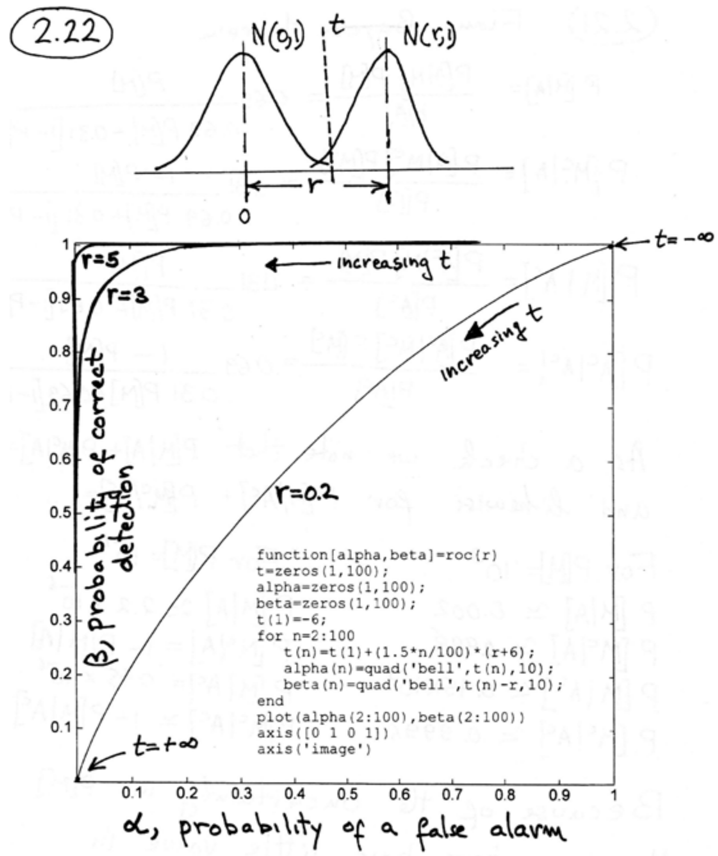
$$\begin{aligned} P[A^c|M] &= P[X < 0.5|M] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}(x-1)^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.5} e^{-\frac{1}{2}y^2} dy, \quad \text{again with } y \triangleq x-1, \\ &= \frac{1}{2} - \text{erf}(0.5) \doteq 0.31. \end{aligned}$$

31. From Bayes' Theorem

$$\begin{aligned} P[M|A] &= \frac{P[A|M]P[M]}{P[A]} = 0.69 \frac{P[M]}{0.69P[M] + 0.31(1 - P[M])}, \\ P[M^c|A] &= \frac{P[A|M^c]P[M^c]}{P[A]} = 0.31 \frac{P[M^c]}{0.69P[M^c] + 0.31(1 - P[M^c])}, \\ P[M|A^c] &= \frac{P[A^c|M]P[M]}{P[A^c]} = 0.31 \frac{P[M]}{0.31P[M] + 0.69(1 - P[M])}, \text{ and} \\ P[M^c|A^c] &= \frac{P[A^c|M^c]P[M^c]}{P[A^c]} = 0.69 \frac{P[M^c]}{0.31P[M] + 0.69(1 - P[M])}. \end{aligned}$$

As a partial check, we note that $P[M|A]+P[M^c|A] = 1$ as it must, and likewise for $P[M|A^c]+P[M^c|A^c]$. Then, for $P[M] = 10^{-3}$, we get $P[M|A] \simeq 2\times10^{-3}$, $P[M^c|A] \simeq 0.998$, $P[M|A^c] \simeq 0.45 \times 10^{-3}$, and $P[M^c|A^c] \simeq 0.9996$. But, for $P[M] = 10^{-6}$, we get $P[M|A] \simeq 2.2 \times 10^{-6}$, $P[M^c|A] \simeq 0.999998$, $P[M|A^c] \simeq 0.45\times10^{-36}$, and $P[M^c|A^c] \simeq 0.999998$. Thus, because of the uncertainty in the prior probability $P[M]$, these calcuated probability numbers have little value for decision making.

32.



A clearer version of the function is given below:

```
function [alpha,beta] = roc(r)
%function for evaluations in Problem 2.32
t=zeros (1,100); alpha=zeros(1,100); beta=zeros(1,100);
t(1)=-6;
for n=2:100
t(n)=t(1)+(1.5*n/100)*(r+6);
alpha(n)=quad('bell',t(n),10);
beta(n)=quad('bell',t(n)-r,10);
end
plot(alpha(2:100),beta(2:100))
axis([0 1 0 1])
axis ('image')
end
```

However, note that this function will only work with definition of the function 'bell', not given here. See documentation on MATLAB function 'quad'.

33. From the data, $\lambda = 9 \times 10^6$ ph/sec. For the counting interval (CI) $\Delta t = 10^{-6}$ sec. then,

$\lambda \Delta t = 9$. So

$$\begin{aligned} P[\{\text{false alarm in CI}\}] &= P[\{\text{ 0 photons in CI}\}] + P[\{\text{ 1 photon in CI}\}] \\ &= \frac{(9)^0}{0!}e^{-9} + \frac{(9)^1}{1!}e^{-9} \\ &= 10e^{-9} \simeq 0.0012. \end{aligned}$$

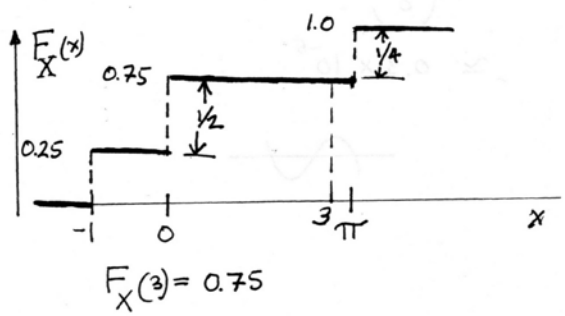
$$\begin{aligned} P[\{\text{at least one false alarm in } 10^6 \text{ tries}\}] &= 1 - P[\{\text{ 0 false alarms in } 10^6 \text{ tries}\}] \\ &\simeq 1 - \binom{10^6}{0}(0.0012)^0(1-0.0012)^{10^6} \\ &= 1 - (0.9988)^{10^6} \\ &\simeq 0. \end{aligned}$$

34. (a) $\Omega = \{G, R, Y\}$, where G=green, R=red, and Y=yellow. The σ-field of events are: $\{G\}, \{R\}, \{Y\}, \{G, R\}$(i.e. light is green or red) , $\{G, Y\}$(i.e.light is green or yellow), $\{R, Y\}$(light is red or yellow) , ϕ(null event), and Ω, the certain event.

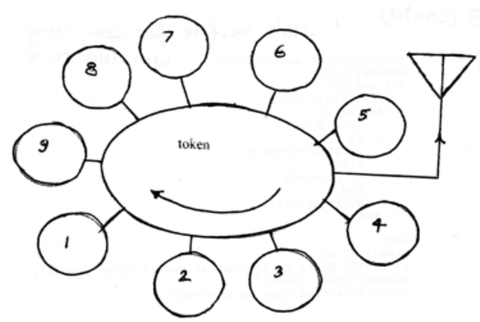
(b) $X(G) = -1, X(R) = 0, X(Y) = \pi$, and $P[G] = P[Y] = 0.5P[R]$. Hence

$$P[R] + 0.5P[R] + 0.5P[R] = 1.$$

So $P[R] = \frac{1}{2}$ and $P[G] = P[Y] = \frac{1}{4}$.



35.



(a)

$$\begin{aligned} T_{\max} &= 8 \times 5 = 40 \text{ minutes}, \\ T_{\min} &= 8 \times 0 = 0 \text{ minutes}. \end{aligned}$$

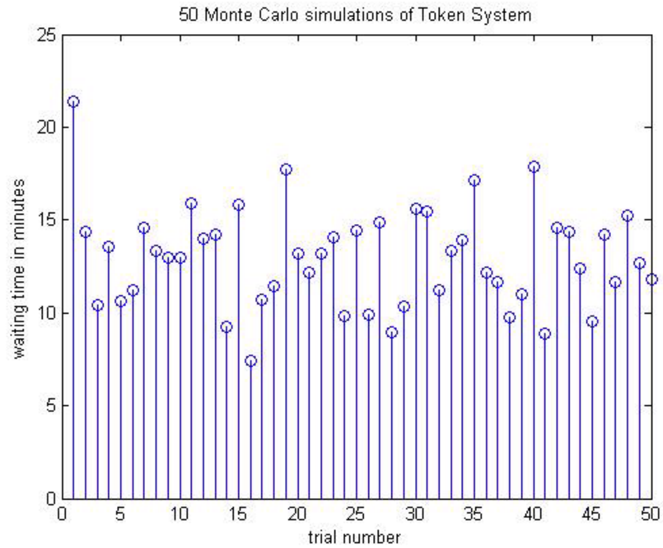
Note that if each station sends a message that is infinitesimally short, you get $T_{\min} = 0$. Let T denote the waiting time and let $p \triangleq P[\{\text{a station is busy}\}]$. Then

$$\begin{aligned} E[T] &= [2.5p + (1-p)1]\,8 \\ &= 8 \text{ minutes for } p = 0, \\ &= 14 \text{ minutes for } p = 0.5, \\ &= 20 \text{ minutes for } p = 1. \end{aligned}$$

(b) Here is a MATLAB function that simulates the waiting time:

```
function [w]=token (p, NT, NS)
% function to simulate the token system in Problem 2.35
% p=probability a station is occupies, NT is number of trials,
% and NS is number of stations.
w=zeros (1, NT);
for n=1:NT
st=zeros(1,9);
for m=2:9
z=rand<=p;
if z>0
st(m)=5*rand;
else
st (m)=1;
end
w(n)=sum (st);
end
end
stem (1: NT, w)
title('50 Monte Carlo simulations of Token System')
xlabel('trial number')
ylabel('waiting time in minutes')
```

Here is a sample output, corresponding to 'probability of station occupied' $p = 0.4$, 'number of trials' $NT = 50$, and 'number of stations' $NS = 9$.



36.

$$\begin{aligned}
P[X^2+Y^2 \le c^2] &= \iint_{(x,y)\ :x^2+y^2\le c^2} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dxdy, \quad \text{transform to polar coordinates} \\
&= \frac{1}{2\pi}\int_0^c \int_0^{2\pi} e^{-\frac{1}{2}r^2} rdrd\theta, \quad \text{with } r=\sqrt{x^2+y^2} \text{ and } dxdy = rdrd\theta, \\
&= \int_0^c e^{-\frac{1}{2}r^2} rdr, \quad \text{let } u \triangleq \frac{1}{2}r^2, \quad \text{then} \quad du = rdr, \\
&= \int_0^{c^2/2} e^{-u} du \\
&= 1 - e^{-c^2/2} = 0.95.
\end{aligned}$$

Thus we need

$$\begin{aligned}
\frac{c^2}{2} &= \ln \frac{1}{1-0.95} = \ln 20 \simeq 3, \\
c &\simeq \sqrt{6} = 2.45.
\end{aligned}$$

37. (a) Since the area of this square with side $\sqrt{2}$ is 2, constant joint density $f_{X,Y}$ must take on value $\frac{1}{2}$ to be properly normalized, thus $A = \frac{1}{2}$.

(b) We can see four regions for the y values in evaluating

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy.$$

These regions are $x \le -1, -1 < x < 0, 0 \le x < 1$,and $x \ge 1$. Now, the first and last of these regions gives the trivial result $f_X(x) = 0$. For $0 \le x < 1$, we get

$$f_X(x) = \int_{x-1}^{1-x} \frac{1}{2}dy = \frac{1}{2}(1-x-x+1) = 1-x.$$

Similarly for $-1 < x < 0$, we get

$$f_X(x) = \int_{-x-1}^{1+x} \frac{1}{2} dy = \frac{1}{2}(1 + x + x + 1) = 1 + x.$$

Combining these regions we finally get

$$f_X(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{else.} \end{cases}$$

(c) If X is close to 1, then we see that Y must be close to 0. This suggests dependence between X and Y. To be sure we can use the result of part b together with the symmetry of the joint density to check whether $f_{X,Y} = f_X f_Y$ or not. By symmetry of $f_{X,Y}$ it must also be that

$$f_Y(y) = \begin{cases} 1 - |y|, & |y| < 1, \\ 0, & \text{else.} \end{cases}$$

Now the product of these two triangles $(1 - |x|)(1 - |y|) \neq \frac{1}{2}$ on $\text{supp}(f_{X,Y})$, so the random variables are definitely dependent. (The *support* of a function $f(x)$ is the set of domain values $\{x|f(x) \neq 0\}$ and is written as $\text{supp}\{f\}$.)

(d) We start with the definition and then plug in our result from part b:

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \begin{cases} \frac{0.5}{1-|x|}, & 0 \leq |x| + |y| < 1, \\ 0, & \text{otherwise in } \{|x| < 1\}, \\ \times, & |x| \geq 1. \end{cases} \end{aligned}$$

Note that the conditional density is not defined for $\{|x| \geq 1\}$.

38. The pdf of the failure time random variable X is

$$\begin{aligned} f_X(t) &= \alpha(t) \exp\left(-\int_0^t \alpha(t')dt'\right) \\ &= \mu \exp(-\mu t) \text{ in this case.} \end{aligned}$$

Assume μ is measured in $(\text{hours})^{-1}$. If $A = \{\text{failure in 100 hrs or less}\}$, then

$$\begin{aligned} P[A] &= P[X \leq 100] \\ &= \int_0^{100} \mu e^{-\mu t} dt \\ &= 1 - e^{-\mu 100} \\ &\leq 0.05? \end{aligned}$$

Thus we need $e^{-\mu 100} \geq 0.95$, or taking logs and solving,

$$\mu \leq 5.13 \times 10^{-4}.$$

39. In general, for any $\alpha(t)$, the pdf of the failure time random variable X is

$$f_X(t) = \alpha(t) \exp\left(- \int_0^t \alpha(t')dt' \right).$$

(i) for $t < 0$, $\alpha(t) = 0$, and so $f_X(t) = 0$,
(ii) for $0 \le t \le 10$, $\alpha(t) = \frac{1}{2}$, and so

$$f_X(t) = \frac{1}{2} e^{-\frac{1}{2}t},$$

(iii) for $t > 10$, $,\alpha(t) = t - c$ for some constant c. Now at $t = 10, \alpha = \frac{1}{2}$, thus $\frac{1}{2} = 10 - c$, and so $c = 9.5$. Then, for $t > 10$,

$$\begin{aligned} f_X(t) &= (t - 9.5) \exp\left(- \left\{ \int_0^{10} \frac{1}{2} dt' + \int_{10}^t (t' - 9.5) dt' \right\} \right) \\ &= (t - 9.5) \exp\left(- \left\{ 5 + (\frac{1}{2}t^2 - 9.5t) - (50 - 95) \right\} \right) \\ &= (t - 9.5) \exp\left(- \left\{ \frac{t^2}{2} - 9.5t + 50 \right\} \right). \end{aligned}$$

We can put all this together in the one equation

$$f_X(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2} e^{-\frac{1}{2}t}, & 0 \le t \le 10 \\ (t - 9.5) \exp\left(- \left\{ \frac{t^2}{2} - 9.5t + 50 \right\} \right), & t > 10. \end{cases}$$

40. (a) Let $\Delta x > 0$ and $\Delta y > 0$, then

$$\begin{aligned} P[x &< X \le x + \Delta x, y < Y \le y + \Delta y] \\ &= F_{XY}(x + \Delta x, y + \Delta y) - F_{XY}(x + \Delta x, y) \\ &\quad - F_{XY}(x, y + \Delta y) + F_{XY}(x, y) \\ &= \left(\frac{F_{XY}(x + \Delta x, y + \Delta y) - F_{XY}(x, y + \Delta y)}{\Delta x} \right) \Delta x \\ &\quad - \left(\frac{F_{XY}(x + \Delta x, y) - F_{XY}(x, y)}{\Delta x} \right) \Delta x \\ &\simeq \left(\frac{\partial F_{XY}(x, y + \Delta y)}{\partial x} - \frac{\partial F_{XY}(x, y)}{\partial x} \right) \Delta x \\ &\simeq \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \Delta x \Delta y \\ &\simeq f_{XY}(x, y) \Delta x \Delta y. \end{aligned}$$

(b) By definition of the density f_{XY} as the mixed partial derivative of the distribution function F_{XY}, the integral of the density over all space must be $F_{XY}(\infty, \infty)$, since $F_{XY}(-\infty, \infty) = F_{XY}(\infty, -\infty) = F_{XY}(-\infty, -\infty)$ are all zero. But, again by definition,

$$\begin{aligned} F_{XY}(\infty, \infty) &= P[X \le \infty, Y \le \infty] \\ &= 1. \end{aligned}$$

(c) From part (a), it follows that

$$f_{XY}(x,y)\Delta x\Delta y \geq 0,$$

since it is a probability. Then since we took $\Delta x > 0$ and $\Delta y > 0$, it follows that

$$f_{XY}(x,y) \geq 0 \quad \text{too.}$$